The Precise Definition of Limit: $\epsilon\delta$-Proofs Dr. Evans

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The precise definition of limit given on page 50 in section 1.8 was developed over hundreds of years. It is therefore no surprise that students often have difficulties quickly mastering the definition and how to use it. The following remarks and examples are meant to serve as a guide as you learn to use use the definition to write your own $\epsilon\delta$-proofs. Please be aware as you study this that styles in proof writing vary greatly, and no two people will ever write the exact same proof. However, the *logical* structure of any two proofs (of the same result) should be more or less the same. In the present setting, you must demonstrate, usually through a series of algebraic steps, that the implication in the definition holds. In an $\epsilon\delta$-proof, you must first do some calculations to find the number δ, but these calculations are *not part of the proof.* Instead, the proof consists of specifying a value for δ in terms of ϵ and showing that the implication in the limit definition holds for this value of δ. Let's begin by making a few remarks about absolute value.

Absolute Value. First recall that if x is a real number, the *absolute value of* x is the distance from x to 0 and is written $|x|$. Said another way, we can define

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Therefore, if c is any real number, we have

$$|x-c| = \begin{cases} x-c & \text{if } x \geq c; \\ c-x & \text{if } x < c \end{cases}$$

so that it is natural (and useful) to think of $|x-c|$ as the distance from x to c. Two important equivalences involving absolute value are

$$|x-c| < \delta \iff -\delta < x-c < \delta \iff c-\delta < x < c+\delta$$

where the symbol $\iff$ means "is equivalent to". In words, these equivalences say that x is less than δ units from c if and only if the difference $x-c$ is between $-\delta$ and δ if and only if x is in the interval $(c-\delta, c+\delta)$. Draw a picture!

The Definition. Let us state the definition of limit, first informally and then precisely.

> **Definition (informal).** *If $f(x)$ is a function defined for all values of x near $x=c$, except perhaps at $x=c$, and if L is a real number such that the values of $f(x)$ get closer and closer to L as the values of x are taken closer and closer to c, then we say L is the limit of $f(x)$ as x approaches c and we write*
>
> $$\lim_{x \to c} f(x) = L.$$

To transform this intuitive idea into a precise definition, we need to say exactly what we mean by " $f(x)$ gets closer and closer to L as the values of x are taken closer and closer to c". The main idea is to notice that if two quantities are getting "closer and closer", then the distance between them is becoming "smaller and smaller". That is, the distance is eventually smaller than any specified positive number. Note that there is an implication in this informal definition. Namely it says *if* we allow x to become closer and closer to c, *then* $f(x)$ will become closer and closer to L. When we write a proof, we show that by taking x sufficiently close to c, we make $f(x)$ arbitrarily close to L. However, before we can demonstrate the implication in the definition, we need to know how close to c is sufficiently close; that is we need to find δ. Now let us state the precise definition.

Definition. *Suppose that c and L are real numbers and $f(x)$ is a function defined in an open interval containing c, except perhaps at $x = c$. If for every positive number $\epsilon > 0$, there exists a positive number $\delta > 0$ (which depends on ϵ) such that*

$$0 < |x - c| < \delta \qquad \text{implies} \qquad |f(x) - L| < \epsilon,$$

then we say that L is the limit of $f(x)$ as x approaches c and we write

$$\lim_{x \to c} f(x) = L.$$

Examples. Now we will write a few proofs to guide you in your own writing. To emphasize the logical structure of the proof, we will not show how we found our δ in the first two examples.

Example 1. *Show that* $\lim_{x \to 2}(3x - 5) = 1$.

Proof. Let $\epsilon > 0$ and define $\delta = \epsilon/3$. Then if $0 < |x - 2| < \delta$, we have

$$\begin{aligned} |(3x-5) - 1| &= |3x - 6| \\ &= 3|x - 2| \\ &< 3(\epsilon/3) && \text{(since } |x-2| < \delta \text{ and } \delta = \epsilon/3) \\ &= \epsilon. \end{aligned}$$

Therefore we have shown

$$0 < |x - 2| < \delta \qquad \text{implies} \qquad |(3x - 5) - 1| < \epsilon,$$

which shows $\lim_{x \to 2}(3x - 5) = 1$ by definition. ∎

Example 2. *Show that* $\lim_{x \to 4}(7x - 1) = 27$.

Proof. Let $\epsilon > 0$ and define $\delta = \epsilon/7$. Then if $0 < |x - 4| < \delta$, we have

$$\begin{aligned} |(7x-1) - 27| &= |7x - 28| \\ &= 7|x - 4| \\ &< 7(\epsilon/7) && \text{(since } |x-4| < \delta \text{ and } \delta = \epsilon/7) \\ &= \epsilon. \end{aligned}$$

Therefore we have shown

$$0 < |x - 4| < \delta \qquad \text{implies} \qquad |(7x - 1) - 27| < \epsilon,$$

which shows $\lim_{x \to 4}(7x - 1) = 27$ by definition. ∎

Each of these examples is a complete proof. However, the question of how we chose the values of δ is unanswered by the proof itself. In fact, some “scratch work” was done before the proof was written. Let’s look at the scratch work now.

Scratch work for Example 1. We want $|(3x-5)-1| < \epsilon$ when $0 < |x-2| < \delta$. We solve the inequality $|(3x-5)-1| < \epsilon$ for $|x-2|$:

$$|(3x-5)-1| < \epsilon \iff |3x-6| < \epsilon \iff 3|x-2| < \epsilon \iff |x-2| < \epsilon/3.$$

The last inequality shows that we should take $\delta = \epsilon/3$ since each step in solving for $|x-2|$ is reversible.

Scratch work for Example 2. We want $|(7x-1)-27| < \epsilon$ when $0 < |x-4| < \delta$. We solve the inequality $|(7x-1)-27| < \epsilon$ for $|x-4|$:

$$|(7x-1)-27| < \epsilon \iff |7x-28| < \epsilon \iff 7|x-4| < \epsilon \iff |x-4| < \epsilon/7.$$

The last inequality shows that we should take $\delta = \epsilon/7$ since each step in solving for $|x-4|$ is reversible.

If the function $f(x)$ is a polynomial of degree $n > 1$, it is often necessary to "condition twice on δ". That is, when performing the algebra involved in the scratch work, we will need to restrict x to bound any extraneous terms in our inequalities. Restricting x is equivalent to keeping it within a certain distance of c, which is again equivalent to choosing a value for δ. Let us make this situation more concrete by way of a specific example. This time, we'll reveal the scratch work before we write the formal proof.

Example 3. *Show that* $\lim_{x\to 5} x^2 = 25$.

Scratch work. We want $|x^2-25| < \epsilon$ when $0 < |x-5| < \delta$. We try to solve the inequality $|x^2-25| < \epsilon$ for $|x-5|$:

$$|x^2-25| < \epsilon \iff |x-5||x+5| < \epsilon \iff |x-5| < \epsilon/|x+5|. \tag{1}$$

This situation differs from **Examples 1** and **2** in that we want to define $\delta = \epsilon/|x+5|$, but we cannot since δ is supposed to be a number depending only on ϵ, not a function of x. Here is one way to get around this difficulty: we will replace $|x+5|$ in (1) by a number M which satisfies $|x+5| \le M$. In so doing, we rewrite (1) as

$$|x^2-25| < \epsilon \iff |x-5||x+5| < \epsilon \iff |x-5|M < \epsilon \iff |x-5| < \epsilon/M \tag{2}$$

and proceed as before taking $\delta = \epsilon/M$. There is a problem here as well. Namely, there is no number M that satisfies $|x+5| \le M$ for all real numbers x. But we are not interested in *all* real numbers x, only those close to $c = 5$. How close? Well, it doesn't matter! We just want to bound $|x+5|$ by restricting x near 5, and any restriction will do. For example, if we require $0 < |x-5| < 1$ (that is x should be less than 1 unit away from 5 or, equivalently $\delta = 1$), then we have

$$|x-5| < 1 \iff -1 < x-5 < 1 \iff 9 < x+5 < 11$$

so that we can take $M = 11$. Now referring back to (2), we should let $\delta = \epsilon/11$. Remember that we also need $|x-5| < 1$ so that if we define $\delta = \min\{1, \epsilon/11\}$, then $0 < |x-5| < \delta$ implies $|x-5| < 1$ and $|x-5| < \epsilon/11$. We can now write the formal proof. Note how much more explanation is present in the scratch work than in the formal proof.

Proof (of Example 3.) Let $\epsilon > 0$ and define $\delta = \min\{1, \epsilon/11\}$. Then if $0 < |x-5| < \delta$, we have

$$\begin{aligned} |x^2-25| &= |x+5||x-5| \\ &< 11|x-5| && \text{(since } |x-5| < \delta \text{ and } \delta \le 1) \\ &< 11(\epsilon/11) && \text{(since } |x-5| < \delta \text{ and } \delta \le \epsilon/11) \\ &= \epsilon. \end{aligned}$$

Therefore we have shown

$$0 < |x-5| < \delta \qquad \text{implies} \qquad |x^2-25| < \epsilon,$$

which shows $\lim_{x\to 5} x^2 = 25$ by definition. ■

Here is one more example, and this time we'll leave the scratch work to the reader. In fact, we suggest that you first do your own scratch work and write your own proof. After this, compare your work to the proof below. Remember, it is unlikely that two proofs will read exactly the same, you need only verify the implication in the definition with your (deduced) value of δ.

Example 4. *Show that* $\lim_{x\to 4}(2x^2 - x + 3) = 31$.

Proof. Let $\epsilon > 0$ and define $\delta = \min\{1, \epsilon/17\}$. Then if $0 < |x-4| < \delta$, we have

$$\begin{aligned}
|(2x^2 - x + 3) - 31| &= |2x^2 - x - 28| \\
&= |2x+7||x-4| \\
&< 17|x-4| && \text{(since } |x-4| < \delta \text{ and } \delta \le 1\text{)} \\
&< 17(\epsilon/17) && \text{(since } |x-4| < \delta \text{ and } \delta \le \epsilon/17\text{)} \\
&= \epsilon.
\end{aligned}$$

Therefore we have shown

$$0 < |x-4| < \delta \qquad \text{implies} \qquad |(2x^2 - x + 3) - 31| < \epsilon,$$

which shows $\lim_{x\to 4}(2x^2 - x + 3) = 31$ by definition. ∎

We will close by giving an example that illustrates the algebra involved when dealing with a rational function. Once again, we'll first show our scratch work involved in finding δ and then write a formal proof.

Example 5. *Show that* $\lim_{x\to 3}\dfrac{1}{x+1} = \dfrac{1}{4}$.

Scratch work. This time, we want $|1/(x+1) - 1/4| < \epsilon$ when $0 < |x-3| < \delta$. Let's try to solve the inequality $|1/(x+1) - 1/4| < \epsilon$ for $|x-3|$:

$$\left|\frac{1}{x+1} - \frac{1}{4}\right| < \epsilon \iff \left|\frac{4-x-1}{(x+1)4}\right| < \epsilon \iff \left|\frac{3-x}{(x+1)4}\right| < \epsilon \iff \frac{|x-3|}{|x+1|} < 4\epsilon. \tag{3}$$

Just like in **Example 3**, we want to find a number M such that

$$\frac{1}{|x+1|} \le M$$

so that we can replace $1/|x+1|$ in (3) with M and proceed to take $\delta = 4\epsilon/M$. Again, there is no M such that $1/|x+1| \le M$ for all x, but if $|x-3| < 1$ (i.e. if $\delta = 1$), then you can show that $3 < x+1 < 5$ so that $1/|x+1| \le 1/3$. Therefore we set $M = 1/3$ and define $\delta = \min\{1, 12\epsilon\}$. It remains to write the proof.

Proof (of Example 5.) Let $\epsilon > 0$ and define $\delta = \min\{1, 12\epsilon\}$. Then if $0 < |x-3| < \delta$, we have

$$\begin{aligned}
\left|\frac{1}{x+1} - \frac{1}{4}\right| &= \left|\frac{4-x-1}{(x+1)4}\right| \\
&= \frac{|x-3|}{|x+1|4} \\
&< \left(\frac{1}{3}\right)\frac{|x-3|}{4} && \text{(since } |x-3| < \delta \text{ and } \delta \le 1\text{)} \\
&< \frac{1}{12}12\epsilon && \text{(since } |x-3| < \delta \text{ and } \delta \le 12\epsilon\text{)} \\
&= \epsilon.
\end{aligned}$$

Therefore we have shown

$$0 < |x-3| < \delta \qquad \text{implies} \qquad \left|\frac{1}{x+1} - \frac{1}{4}\right| < \epsilon,$$

which shows $\lim_{x\to 3}[1/(x+1)] = 1/4$ by definition. ■

Exercise 1. Show that $\lim_{x\to 3}(2x-1) = 5$.

Exercise 2. Show that $\lim_{x\to 0}(x+7) = 7$.

Exercise 3. Show that $\lim_{x\to -1}(3x+2) = -1$.

Exercise 4. Show that $\lim_{x\to 2}(x^2+x) = 6$.

Exercise 5. Show that $\lim_{x\to 1}(x^2+3x+2) = 6$.